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Inhomogeneous electric field generated by two long electrodes placed along parallel infinite walls separating different dielectric media

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Abstract. Analytical solutions for the electric potential in different mixed-boundary-value problems are found by use of the Wiener-Hopf method. The problem arises in the investigation of the influence of a non-uniform electric field on an electrorheological fluid flowing through a plane channel. The solution is given in terms of infinite series involving Gamma functions. The electric field in the vicinity of the electrode edges is evaluated asymptotically. Some parametric studies are made with respect to the ratio between the permittivity of the electrorheological fluid and the permittivity of the isolating material outside the channel. Numerical solutions are also given.

Key words: jump conditions, nonuniform electric field, semi-infinite and finite electrode, Wiener-Hopf technique

1. Introduction

We determine here the distribution of the electric potential around two long electrodes charged with different potentials in a symmetric, an anti-symmetric and a non-symmetric way. The term "long electrode" denotes here either a semi-infinite electrode or a finite electrode of a certain length chosen so that the two far edges of the electrode do not interact. The final goal is to use these solutions in further studies concerning the channel flow of electrorheological fluids (ERF). These are fluids that change their material properties (especially the viscosity) if they are exposed to an electric field (see [1]). In modeling the ERF it is assumed, in a first approximation, that the flow does not affect the electric field. Consequently, the electric problem can be decoupled from the mechanical equations and the study of the solution for the electric field can be carried out independently and then used in a second step in the mechanical problem. In this paper we study the electrical problem as a preparation to the flow problem determined in a second step.

So far, in many works concerning channel flows of electrorheological fluids (see [2] and [3]), the assumption of a homogeneous electric field has been used in order to calculate analytic solutions for the velocity field, by considering that the electrodes are infinite. In reality, important inhomogeneity effects appear in the vicinity of the electrode edges which have not yet been modeled. In addition, experimental investigations (see [4] and [5]) demonstrated that inhomogeneities in the electric field have a greater effect on the flow than an-

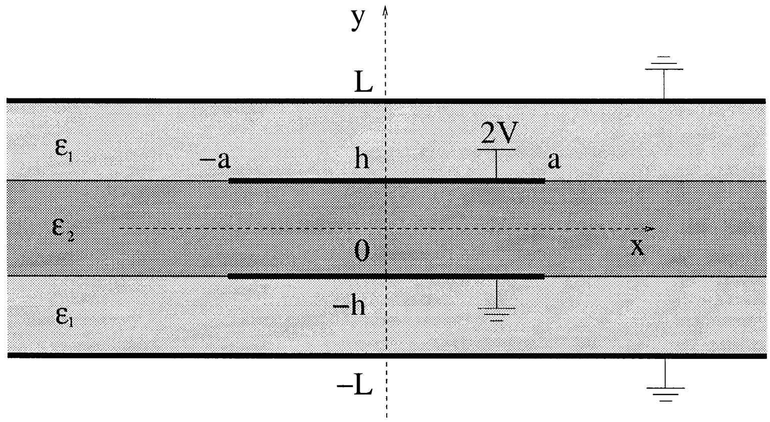


Figure 1. Configuration of the electrodes with non-symmetric boundary conditions.

ticipated. This led to the idea that more-efficient effects for applications could be obtained under strongly space-dependent electric fields. Here we present (semi)-analytical solutions by using the Wiener-Hopf technique. The reason for doing this is two-fold: on the one hand, the method of construction allows us to find an entire class of solutions for technically interesting configurations, and, on the other hand, evaluation of the electric and velocity fields for these solutions via the WH-technique, is computationally far more economical (the CPU-times of the latter are a factor of 50–100 larger than for the former) than using the available numerical integrators for elliptic equations for the same configurations. Furthermore, the WH-technique features the singularities at the tips of the electrodes explicitly, which in a numerical solution must be approximately accounted for by rather costly mesh refinements.

In Section 2, the general problem for the electric potential is formulated as a mixed-boundary-value problem. Continuity conditions in the tangential derivative and jump conditions in the normal derivative are taken into account across the channel walls. Owing to the linearity of the problem, the solution is constructed as a sum of two solutions for two particular problems: with symmetric and anti-symmetric boundary conditions. The Wiener-Hopf procedure is successfully applied to solve these problems in Sections 3 and 4. First we deduce the Wiener-Hopf equations and then the factorizations are accomplished. The singular behaviour of the electric field near the ends of the electrodes is determined in Section 5. In Section 6 the results are plotted and some remarks are presented. The final section is devoted to summarizing the results.

2. Formulation of the problem

Let Oxy be a Cartesian coordinate system. We consider an infinitely long channel of height $2h$ consisting of two infinite parallel planes of zero thickness (see Figure 1). These planes are situated at $y = -h$ and $y = h$, respectively. Along the channel walls, two electrodes charged with different potentials are placed (at the upper electrode a constant potential $2V$ is applied and the lower electrode is grounded). The electrodes are insulated outside the channel by a dielectric material having electric permittivity ε_1. The medium inside the channel has electric permittivity ε_2. We solve the problem in a bounded domain in the y-direction, where the upper and lower boundaries consist of two infinite grounded electrodes situated at a distance $L > h$ from the x-axis. If L is sufficiently large, this is equivalent with the usual conditions at infinity requiring the potential to vanish at $y = \pm\infty$. For smaller L, this configuration can still be easily realized in practice. We mention that the two grounding electrodes at $y = \pm L$ are needed for technical reasons when solving the WH-problem, for otherwise no solution could be found, *i.e.*, when no grounding electrodes are present. Physically, this is no restriction because the channel will always be earthed and the system can always be considered for large L.

First, we will formulate and solve the problem in the case of semi-infinite electrodes. The origin of the coordinate system is shifted to the left by an amount $x = a$ and the electrodes are prolonged to infinity on the right side. The solution can be used directly to describe the corresponding case with finite electrodes provided that their length allows non-coupling of the left and right electrode ends. From parametric studies we can determine such length values in relation with L, h and $c = \varepsilon_1/\varepsilon_2$ (see Section 6). In summary our problem reads

$$\nabla^2\varphi = 0 \quad \text{in} \quad -\infty < x < \infty, \quad -L \leq y \leq L\,, \tag{2.1}$$

with the boundary conditions

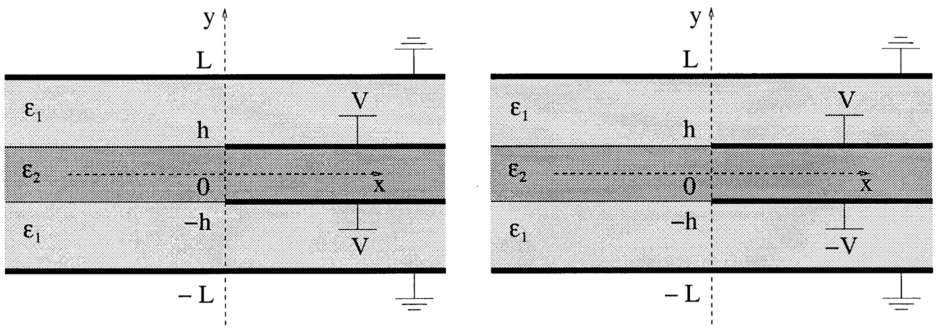
$$\varphi\,(x, h) = 2V, \quad x \geq 0\,, \tag{2.2}$$
$$\varphi\,(x, -h) = 0, \quad x \geq 0\,, \tag{2.3}$$
$$\varphi_{,1}\,(x,\, h^+) = \varphi_{,1}\,(x,\, h^-), \quad x \leq 0\,, \tag{2.4}$$
$$\varphi_{,1}\,(x,\, -h^+) = \varphi_{,1}\,(x,\, -h^-), \quad x \leq 0\,, \tag{2.5}$$
$$\varphi_{,2}\,(x,\, h^-) = c\,\varphi_{,2}\,(x,\, h^+), \quad x \leq 0\,, \tag{2.6}$$
$$\varphi_{,2}\,(x,\, -h^+) = c\,\varphi_{,2}\,(x,\, -h^-), \quad x \leq 0\,, \tag{2.7}$$
$$\varphi(x, \pm L) = 0, \quad -\infty < x < \infty\,. \tag{2.8}$$

Here the quantity φ denotes the electric potential. The electric field is then given by $E = -\text{grad}\,\varphi$. The notations $\varphi_{,1}$, $\varphi_{,2}$ are used to denote the partial derivatives of the function φ with respect to x and y respectively. To specify the jump of the components of the electric field across $y = \pm h$ for $x \leq 0$, we use the upper index "$+$" to indicate the limit as y tends to $\pm h$ from positive values of $(y \mp h)$ and the upper index "$-$" to indicate the limit as y tends to $\pm h$ from negative values of $(y \mp h)$. Since the domain is symmetric with respect to the x-axis, it is more convenient to split the problem, by applying the principle of superposition, in two easier problems corresponding to the symmetric and anti-symmetric parts of the unknown function φ with respect to y (see Figure 2):

$$\varphi^s(x, y) = \frac{\varphi(x, y) + \varphi(x, -y)}{2}\,, \tag{2.9}$$
$$\varphi^a(x, y) = \frac{\varphi(x, y) - \varphi(x, -y)}{2}\,. \tag{2.10}$$

Again, from symmetry considerations, it is sufficient to solve these two problems for $y \geq 0$. It is easy to prove that the boundary-value problem (BVP) for the symmetric part reads

$$\nabla^2\varphi^s = 0 \quad \text{in} \quad -\infty < x < \infty, \quad 0 \leq y \leq L\,, \tag{2.11}$$



a. Symmetric case b. Anti-symmetric case

Figure 2. Schematic diagram of the electrodes configuration for the particular problems.

where φ^s satisfies the boundary conditions

$$\varphi^s\,(x, h) = V, \quad x \geq 0\,, \tag{2.12}$$
$$\varphi^s_{,2}\,(x,\,0) = 0, \quad -\infty < x < \infty\,, \tag{2.13}$$
$$\varphi^s_{,1}\,(x,\,h^+) = \varphi^s_{,1}\,(x,\,h^-), \quad x \leq 0\,, \tag{2.14}$$
$$\varphi^s_{,2}\,(x,\,h^-) = c\,\varphi^s_{,2}\,(x,\,h^+), \quad x \leq 0\,, \tag{2.15}$$
$$\varphi^s(x, L) = 0, \quad -\infty < x < \infty\,, \tag{2.16}$$

while the BVP for the anti-symmetric part is

$$\nabla^2\varphi^a = 0 \quad \text{in} \quad -\infty < x < \infty, \quad 0 \leq y \leq L\,, \tag{2.17}$$

with the boundary conditions

$$\varphi^a\,(x, h) = V, \quad x \geq 0\,, \tag{2.18}$$
$$\varphi^a\,(x, 0) = 0, \quad -\infty < x < \infty\,, \tag{2.19}$$
$$\varphi^a_{,1}\,(x,\,h^+) = \varphi^a_{,1}\,(x,\,h^-), \quad x \leq 0\,, \tag{2.20}$$
$$\varphi^a_{,2}\,(x,\,h^-) = c\,\varphi^a_{,2}\,(x,\,h^+), \quad x \leq 0\,, \tag{2.21}$$
$$\varphi^a(x, L) = 0, \quad -\infty < x < \infty\,. \tag{2.22}$$

We simply remark that, so far, the two problems differ only through the conditions (2.13) and (2.19) required by symmetry and anti-symmetry, respectively. We continue by solving them in parallel and writing the common relations only once (with double superscript).

3. Method of solution

We will solve the afore-mentioned problem by following the method given by Noble [6]. To avoid the difficulties that appear when Fourier transforming a constant function, we will replace $\varphi^{s/a}(x, h) = V$ by

$$\varphi^{s/a}(x, h) = V\mathrm{e}^{-\varepsilon x} =: \varphi_0(x)\,, \quad 0 \leq x < \infty \quad (\varepsilon > 0). \tag{3.1}$$

We shall ultimately let $\varepsilon \to 0$.[1] Multiplying the Laplace equation by $\mathrm{e}^{\mathrm{i}\alpha x}$, with α being the Fourier transform variable, and integrating the resulting equation with respect to x from $-\infty$

[1] The idea of 3.1 has been adopted from [6, pp. 135].

to ∞, we obtain

$$\mathrm{d}^2\Phi^{s/a}(\alpha, y)/\mathrm{d}y^2 - \alpha^2\Phi^{s/a}(\alpha, y) = 0\ , \quad 0 \le y \le L\ , \tag{3.2}$$

where

$$\Phi^{s/a}(\alpha, y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \varphi^{s/a}\, \mathrm{e}^{\mathrm{i}\alpha x}\, \mathrm{d}x\ . \tag{3.3}$$

The solution of (3.2) is

$$\Phi^{s/a}(\alpha, y) = \begin{cases} A_1^{s/a}(\alpha)\, \mathrm{e}^{-\alpha y} + B_1^{s/a}(\alpha)\, \mathrm{e}^{\alpha y}\ , & 0 \le y \le h\ , \\ A_2^{s/a}(\alpha)\, \mathrm{e}^{-\alpha y} + B_2^{s/a}(\alpha)\, \mathrm{e}^{\alpha y}\ , & h \le y \le L\ . \end{cases} \tag{3.4}$$

The Fourier-transformed common boundary conditions are

$$\Phi_+^{s/a}(\alpha, h^+) = \Phi_+^{s/a}(\alpha, h^-) = \Phi_0, \qquad \Phi_-^{s/a}(\alpha, h^+) = \Phi_-^{s/a}(\alpha, h^-), \tag{3.5, 6}$$

$$(\Phi_-^{s/a})'(\alpha, h^-) = c\, (\Phi_-^{s/a})'(\alpha, h^+) \qquad \Phi_-^{s/a}(\alpha, L) = \Phi_+^{s/a}(\alpha, L) = 0, \tag{3.7, 8}$$

where

$$\Phi_+^{s/a}(\alpha, y) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \varphi^{s/a}\, \mathrm{e}^{\mathrm{i}\alpha x}\, \mathrm{d}x\ , \tag{3.9}$$

$$\Phi_-^{s/a}(\alpha, y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \varphi^{s/a}\, \mathrm{e}^{\mathrm{i}\alpha x}\, \mathrm{d}x\ , \tag{3.10}$$

$$\Phi_0(\alpha) = k/(\varepsilon - \mathrm{i}\alpha)\ \ ,\ \ k = V/\sqrt{2\pi}\ , \tag{3.11}$$

and where the dash is used to denote differentiation with respect to y. In addition, (2.13) and (2.19) yield

$$(\Phi_+^s)'(\alpha, 0) = (\Phi_-^s)'(\alpha, 0) = 0, \qquad \Phi_+^a(\alpha, 0) = \Phi_-^a(\alpha, 0) = 0. \tag{3.12, 13}$$

From now on, the α-argument of the functions will be dropped. Using (3.4), (3.12), (3.8) and $\Phi^s(\alpha, h^+) = \Phi^s(\alpha, h^-)$ by (3.5) and (3.6), we deduce

$$\Phi^s(\alpha, y) = \begin{cases} +2A_1^s\ \cosh(\alpha y)\ , & 0 \le y \le h\ , \\ +2A_1^s \dfrac{\cosh(\alpha h)\sinh(\alpha L - \alpha y)}{\sinh(\alpha L - \alpha h)}\ , & h \le y \le L\ . \end{cases} \tag{3.14}$$

After straightforward calculations we obtain from (3.14), on using (3.5–3.12), the relations

$$\Phi_0 + \Phi_-^s(h) = 2A_1^s\ \cosh(\alpha h), \tag{3.15}$$

$$(\Phi_+^s)'(h^+) + (\Phi_-^s)'(h^+) = -2\alpha\, A_1^s\ \cosh(\alpha h)\coth(\alpha(L - h)), \tag{3.16}$$

$$(\Phi^s_+)'(h^-) + c(\Phi^s_-)'(h^+) = 2\alpha\, A^s_1\ \sinh(\alpha h). \tag{3.17}$$

Analogously, (3.4), (3.13), (3.8) and $\Phi^a(\alpha, h^+) = \Phi^a(\alpha, h^-)$, by (3.5) and (3.6), yield

$$\Phi^a(\alpha, y) = \begin{cases} -2A^a_1\ \sinh(\alpha y)\ , & 0 \le y \le h\ , \\ -2A^a_1 \dfrac{\sinh(\alpha h)\sinh(\alpha L - \alpha y)}{\sinh(\alpha L - \alpha h)}\ , & h \le y \le L\ . \end{cases} \tag{3.18}$$

Then from (3.18) and (3.5–3.13) we have

$$Phi_0 + \Phi^a_-(h) = -2A^a_1\ \sinh(\alpha h), \tag{3.19}$$

$$(\Phi^a_+)'(h^+) + (\Phi^a_-)'(h^+) = 2\alpha\, A^a_1\ \sinh(\alpha h)\coth(\alpha(L-h)), \tag{3.20}$$

$$(\Phi^a_+)'(h^-) + c(\Phi^a_-)'(h^+) = -2\alpha\, A^a_1\ \cosh(\alpha h). \tag{3.21}$$

This puts us now in the position to formulate the Wiener-Hopf equation. To this end, multiply (3.16) and (3.20) by c and subtract the resulting equations from (3.17) and (3.21), respectively. If we introduce $F^{s/a}_+ = c\,(\Phi^{s/a}_+)'(h^+) - (\Phi^{s/a}_+)'(h^-)$ and $F^{s/a}_- = \Phi^{s/a}_-(h)$, we obtain, by eliminating the unknown coefficient $A^{s/a}_1$, the Wiener-Hopf equations for the unknown functions $F^{s/a}_+$ and $F^{s/a}_-$ corresponding to each problem, namely

$$K^{s/a}F^{s/a}_+ + F^{s/a}_- = -\Phi_0\ , \tag{3.22}$$

where

$$K^s(\alpha) = \frac{1}{\alpha\ [\ c\coth(\alpha L - \alpha h) + \tanh(\alpha\ h)\ ]}\ , \tag{3.23}$$

$$K^a(\alpha) = \frac{1}{\alpha\ [\ c\coth(\alpha L - \alpha h) + \coth(\alpha\ h)\ ]}\ . \tag{3.24}$$

If Equation (3.22) is solved, the coefficients $A^{s/a}_1$ are found by inserting the functions $F^{s/a}_-$ in (3.15) and in (3.19), respectively. We can obtain the Fourier transforms of the solutions from (3.14) and (3.18) and then, by inverting them, we obtain the final solutions.

4. Application of the Wiener-Hopf technique

In addition to the boundary conditions from Section 2, some regularity assumptions concerning the potential function are needed in order to ensure the applicability of the Wiener-Hopf technique. Assuming that $\varphi(x, h)$ is a bounded function of x for $x \le 0$, we have that the function $F^{s/a}_-$ will be analytic in the half-plane $\tau < 0$, $-\infty < \sigma < \infty$, where $\alpha = \sigma + \mathrm{i}\tau$. It is also reasonable, because of (3.1), to expect that $\varphi(x, y)$ decays exponentially to zero as $x \to \infty$, *i.e.*, there exists a $b > 0$ such that $\varphi(x, y)\mathrm{e}^{-bx}$ is absolutely integrable over the positive x-axis for all y. This yields that $(\Phi^{s/a}_+)'(h^+)$ and $(\Phi^{s/a}_+)'(h^-)$ are analytic for $\tau > -b$. We may take $b = \varepsilon$, where ε is the small parameter introduced in (3.1). Now, the functions $F^{s/a}_-$ and $F^{s/a}_+$ are both analytic in the strip $-\varepsilon < \tau < 0$. If a decomposition of the kernel $K^{s/a}$ can be accomplished in the form $K^{s/a}(\alpha) = K^{s/a}_-(\alpha)K^{s/a}_+(\alpha)$ where $K^{s/a}_+$ is analytic and non-zero for $\tau > -\varepsilon$, and $K_-^{s/a}$ is analytic and non-zero for $\tau < 0$, one may rearrange (3.22) as

$$K_+^{s/a} F_+^{s/a} + F_-^{s/a} / K_-^{s/a} = -\Phi_0 / K_-^{s/a} \, . \tag{4.1}$$

Writing

$$-\Phi_0 / K_-^{s/a} = H_+^{s/a} + H_-^{s/a} \, , \tag{4.2}$$

where $H_-^{s/a}(\alpha)$ and $H_+^{s/a}(\alpha)$ are analytic in $\tau < 0$ and $\tau > -\varepsilon$, respectively, we may define a new function,

$$J^{s/a}(\alpha) \; : \; = K_+^{s/a}(\alpha) F_+^{s/a}(\alpha) - H_+^{s/a}(\alpha) = -F_-^{s/a}(\alpha)/K_-^{s/a}(\alpha) + H_-^{s/a}(\alpha) \, , \tag{4.3}$$

in $-\varepsilon < \tau < 0$. Because of the properties of the second and third parts of this equation, by analytic continuation, $J(\alpha)$ can be defined over the whole α-plane as an entire function. Using the order properties of the functions $K_+^{s/a} F_+^{s/a} - H_+^{s/a}$ and $-F_-^{s/a}/K_-^{s/a} + H_-^{s/a}$ for large values of α, one can determine the form of $J^{s/a}(\alpha)$ with the help of the Liouville theorem and then find $F_-^{s/a}(\alpha)$.

The most important step in the solution consists in decomposing the Wiener-Hopf kernel $K^{s/a}(\alpha)$. This can be accomplished by inspection when infinite-product representations of the numerator and denominator are known. We shall derive these representations as follows: first an appropriate form of $K^{s/a}$ is needed,

$$K^s(\alpha) = \frac{[2 \sinh(\alpha L - \alpha h) \, \cosh(\alpha h)]/\alpha}{[\, (c+1)\cosh(\alpha L) + (c-1)\cosh(\alpha L - 2\alpha h) \,]} \, , \tag{4.4}$$

$$K^a(\alpha) = \frac{[2 \sinh(\alpha L - \alpha h) \, \sinh(\alpha h)]/\alpha^2}{[\, (c+1)\sinh(\alpha L) + (1-c)\sinh(\alpha L - 2\alpha h) \,]/\alpha} \, . \tag{4.5}$$

Now, we can decompose the numerator and the denominator of $K^{s/a}$, using the infinite-product theorem applied for an even function (see [6, pp. 15 and pp. 40]). To do this, the zeros of the functions should be determined. The numerator vanishes for $\alpha = \pm \mathrm{i}\kappa_n$ and $\alpha = \pm \mathrm{i}\lambda_n$ in the symmetric case, and for $\alpha = \pm \mathrm{i}\mu_n$ and $\alpha = \pm \mathrm{i}\kappa_n$ in the anti-symmetric case, where $\kappa_n = n\pi/(L-h)$, $\lambda_n = (n - 1/2)\pi/h$ and $\mu_n = n\pi/h$, $(n = 1, 2, 3, ...)$. It is easy to prove that the zeros of the denominator are purely imaginary in both cases [7]. Consequently, the zeros $\alpha = \mathrm{i}\tau$ can be found by solving the real equations (with τ as unknown)

$$\cos(\tau L) = c_1 \, \cos(\tau(L - 2h)) \, , \qquad \sin(\tau L) = -c_1 \, \sin(\tau(L - 2h)) \, , \tag{4.6, 7}$$

where $c_1 = (1 - c)/(1 + c)$, $|c_1| < 1$. There are two special cases when the roots of (4.6) and (4.7) are explicitly known: when $L = 2h$ and when $\varepsilon_1 = \varepsilon_2$ that means $c = 1$ and $c_1 = 0$. Except for these particular situations, Equations (4.6) and (4.7) cannot be solved analytically and, consequently, their zeros must be found numerically. Now, let us study the periodicity of the solutions. Suppose that $L > 2h$ and that $L/(L - 2h)$ is a rational number written as

$$L/(L - 2h) = p/q \, , \tag{4.8}$$

where p, q are positive integers that are mutually prime, with $p > q$. Then it follows that the complete solutions of (4.6) and (4.7) are $\tau = \pm\delta_{nl} = \pm(\delta_l + 2p\pi(n-1)/L)$ and $\tau = 0$, $\tau = \pm\gamma_{nl} = \pm(\gamma_l + 2p\pi(n-1)/L)$, respectively, $l = 1, 2, ... 2p$, $n = 1, 2, 3...$, where δ_l and

γ_l are the solutions of these equations only in the interval $(0, 2\pi p/L]$, ordered increasingly [7]. So, the zeros of the denominators are $\alpha = \pm \mathrm{i}\,\delta_{nl}$ in the symmetric case and $\alpha = \pm \mathrm{i}\,\gamma_{nl}$ in the anti-symmetric one. We note that δ_l is located between $(l-1)\pi/L$ and $l\pi/L$, while γ_l is located between $(2l-1)\pi/2L$ and $(2l+1)\pi/2L$ and $\gamma_{2p} = 2\pi p/L$. Moreover, if δ_l and γ_l are solutions of (4.6) and (4.7), respectively, in the interval $(0, 2\pi p/L]$ then $2\pi p/L - \delta_l$ and $2\pi p/L - \gamma_l$ are also solutions of (4.6) and (4.7), respectively. δ_l and γ_l will be numerically determined with good precision using the MATHEMATICA software [8].

We mention that the choice of L and h (4.8) is not restrictive for the concrete model but very advantageous since it reduces the computation only to the interval $(0, 2\pi p/L]$. Moreover, (4.8) provides us with the approximate locations of the zeros and this is helpful in evaluating the asymptotic behaviour of the split functions of $K^s(\alpha)$ and $K^a(\alpha)$. Applying the infinite product theorem, we finally arrive at the representations

$$K^s(\alpha) = \frac{2(L-h)\prod_{n=1}^{\infty}\left\{\left[1+\left(\frac{\alpha}{\lambda_n}\right)^2\right]\left[1+\left(\frac{\alpha}{\kappa_n}\right)^2\right]\right\}}{2c\prod_{n=1}^{\infty}\left\{\prod_{l=1}^{2p}\left[1+\left(\frac{\alpha}{\delta_{nl}}\right)^2\right]\right\}}, \tag{4.9}$$

$$K^a(\alpha) = \frac{2h(L-h)\prod_{n=1}^{\infty}\left\{\left[1+\left(\frac{\alpha}{\mu_n}\right)^2\right]\left[1+\left(\frac{\alpha}{\kappa_n}\right)^2\right]\right\}}{[\,2L-2h(1-c)\,]\prod_{n=1}^{\infty}\left\{\prod_{l=1}^{2p}\left[1+\left(\frac{\alpha}{\gamma_{nl}}\right)^2\right]\right\}}. \tag{4.10}$$

The last two formulas can also be applied to the particular cases $c = 1$ and $L = 2h$. However, in these cases p will not be determined from (4.8); in fact, it can be proved that p can be assigned any positive integer value larger than 1 and so we will take $p = 2$ if $c = 1$ or $L = 2h$.

We can now write

$$K^{s/a}(\alpha) = K_+^{s/a}(\alpha) K_-^{s/a}(\alpha)\,, \tag{4.11}$$

where

$$K_\pm^s(\alpha) \;=\; c_2^s\,\mathrm{e}^{\pm\chi^s(\alpha)}\,\frac{\prod_{n=1}^{\infty}\left\{\left(1\mp\frac{\mathrm{i}\alpha}{\lambda_n}\right)\mathrm{e}^{\pm\mathrm{i}\alpha/\lambda_n}\left(1\mp\frac{\mathrm{i}\alpha}{\kappa_n}\right)\mathrm{e}^{\pm\mathrm{i}\alpha/\kappa_n}\right\}}{\prod_{n=1}^{\infty}\left\{\prod_{l=1}^{2p}\left(1\mp\frac{\mathrm{i}\alpha}{\delta_{nl}}\right)\mathrm{e}^{\pm\mathrm{i}\alpha/\frac{2p\pi n}{L}}\right\}}, \tag{4.12}$$

$$K_\pm^a(\alpha) \;=\; c_2^a\,\mathrm{e}^{\pm\chi^a(\alpha)}\,\frac{\prod_{n=1}^{\infty}\left\{\left(1\mp\frac{\mathrm{i}\alpha}{\mu_n}\right)\mathrm{e}^{\pm\mathrm{i}\alpha/\mu_n}\left(1\mp\frac{\mathrm{i}\alpha}{\kappa_n}\right)\mathrm{e}^{\pm\mathrm{i}\alpha/\kappa_n}\right\}}{\prod_{n=1}^{\infty}\left\{\prod_{l=1}^{2p}\left(1\mp\frac{\mathrm{i}\alpha}{\gamma_{nl}}\right)\mathrm{e}^{\pm\mathrm{i}\alpha/\frac{2p\pi n}{L}}\right\}}, \tag{4.13}$$

with

$$c_2^s = \sqrt{\frac{L-h}{c}}\,, \quad c_2^a = \sqrt{\frac{h(L-h)}{L-h(1-c)}}\,. \tag{4.14}$$

The functions $\chi^{s/a}(\alpha)$ are arbitrary and have to be chosen to ensure that $K_+^{s/a}$ and $K_-^{s/a}$ have simple asymptotic behaviour as $|\alpha| \to \infty$ in the appropriate half-planes. To remove the infinite products it is convenient to express the functions $K_+^{s/a}$ and $K_-^{s/a}$ in terms of Γ-functions by use of the well-known formula (see [6, pp. 41])

$$\prod_{n=1}^{\infty}(1+\frac{\alpha}{an+b})\mathrm{e}^{-\alpha/(an)} = \mathrm{e}^{-C\alpha/a}\Gamma(\frac{b}{a}+1)/\,\Gamma(\frac{\alpha}{a}+\frac{b}{a}+1)\,, \tag{4.15}$$

where $C = 0{\cdot}5772\ldots$ is the Euler constant. This leads to the representations

$$K_{\pm}^{s}(\alpha) \;=\; \frac{c_2^s}{c_5^s}\,\Gamma\left(\frac{1}{2}\right) \frac{\prod_{l=1}^{2p}\Gamma\left[\mp\frac{\mathrm{i}\alpha L}{2p\pi}+\frac{\delta_l L}{2p\pi}\right]\exp[\pm\chi^s(\alpha)\pm\mathrm{i}\alpha h\frac{2\log 2}{\pi}]}{\Gamma\left[1\mp\mathrm{i}\frac{\alpha(L-h)}{\pi}\right]\Gamma\left(\frac{1}{2}\mp\mathrm{i}\frac{\alpha h}{\pi}\right)}, \tag{4.16}$$

$$K_{\pm}^{a}(\alpha) \;=\; \frac{c_2^a}{c_5^a}\,\frac{\prod_{l=1}^{2p}\Gamma\left[\mp\frac{\mathrm{i}\alpha L}{2p\pi}+\frac{\gamma_l L}{2p\pi}\right]\exp[\pm\chi^a(\alpha)]}{\Gamma\left[1\mp\mathrm{i}\frac{\alpha(L-h)}{\pi}\right]\Gamma\left(1\mp\mathrm{i}\frac{\alpha h}{\pi}\right)}, \tag{4.17}$$

where $c_5^s = \prod_{l=1}^{2p}\Gamma\left[\frac{\delta_l L}{2p\pi}\right]$ and $c_5^a = \prod_{l=1}^{2p}\Gamma\left[\frac{\gamma_l L}{2p\pi}\right]$. Employing Stirling's formula [9, pp. 257], we find the asymptotic forms

$$K_{\pm}^{s}(\alpha) \;\sim\; B^s\exp[\pm\chi^s(\alpha)\pm\mathrm{i}\alpha h\frac{2\log 2}{\pi}]\,(\mp\mathrm{i}\alpha)^{c_3^s}(c_4)^{\mp\mathrm{i}\alpha}\;,\;\text{ as }\quad |\alpha|\to\infty\,, \tag{4.18}$$

$$K_{\pm}^{a}(\alpha) \;\sim\; B^a\exp[\pm\chi^a(\alpha)]\,(\mp\mathrm{i}\alpha)^{c_3^a}(c_4)^{\mp\mathrm{i}\alpha}\;,\quad\text{as}\quad |\alpha|\to\infty\,, \tag{4.19}$$

where $B^{s/a}$ are constants independent of α and

$$c_3^{s/a} \;=\; -\frac{1}{2}\,, \tag{4.20}$$

$$c_4 \;=\; \left(\frac{L}{2p(L-h)}\right)^{L/\pi}\left(\frac{L-h}{h}\right)^{h/\pi}\;. \tag{4.21}$$

To get a simple asymptotic behaviour of $K_{\pm}^{s/a}(\alpha)$ as $|\alpha|\to\infty$, we choose

$$\chi^s(\alpha) = \mathrm{i}\,\alpha\,\log c_4 - \mathrm{i}\,\alpha\,h\tfrac{2\log 2}{\pi}\,,\qquad \chi^a(\alpha) = \mathrm{i}\,\alpha\,\log c_4\,. \tag{4.22, 23}$$

By inserting the expressions of $\chi^s(\alpha)$, $\chi^a(\alpha)$ in (4.16) and (4.17), respectively, we arrive at the final representations for $K_{\pm}^s$, $K_{\pm}^a$.

If, moreover, $\varepsilon < \min\,(\,\gamma_1,\;\delta_1,\;\kappa_1,\;\lambda_1\,)$, it can be easily checked that $K_-^{s/a}(\alpha)$ is analytic for $\tau < \varepsilon$ and $K_+^{s/a}(\alpha)$ is analytic for $\tau > -\varepsilon$, and, consequently, $K^{s/a}(\alpha)$ is analytic for $-\varepsilon < \tau < \varepsilon$. We decompose the function on the left-hand side of (4.2) as follows

$$\begin{aligned} -\frac{\Phi_0}{K_-^{s/a}(\alpha)} \;&=\; \frac{-k}{(\varepsilon-\mathrm{i}\alpha)K_-^{s/a}(-\mathrm{i}\varepsilon)}+\frac{-k}{\varepsilon-\mathrm{i}\alpha}\left[\frac{1}{K_-^{s/a}(\alpha)}-\frac{1}{K_-^{s/a}(-\mathrm{i}\varepsilon)}\right] \\ &=\; H_+^{s/a}(\alpha)+H_-^{s/a}(\alpha)\,, \end{aligned} \tag{4.24}$$

where $H_-(\alpha)$ and $H_+(\alpha)$ are analytic in $\tau < \varepsilon$ and $\tau > -\varepsilon$, respectively. Let us now return to Equation (4.3); we wish to determine the functions $J^{s/a}(\alpha)$. We have already shown that $K_-^{s/a}$, $K_+^{s/a}$ are asymptotic to $|\alpha|^{-1/2}$ as $|\alpha| \to \infty$. Next, some specific assumptions about the behaviour of $\varphi^{s/a}(x, y)$ in the vicinity of $x = 0$ are necessary, in order to be able to use the Abelian theorem for the Fourier transform for finding the properties of $F_+^{s/a}$ and $F_-^{s/a}$ as $|\alpha| \to \infty$. We assume that $\varphi^{s/a}(x, h) = O(1)$, as $x \to 0$ with $x < 0$, and $\partial\varphi^{s/a}/\partial y\,(x, h^{\pm}) = O(x^{-1/2})$, as $x \to 0$ with $x > 0$. This implies that $F_-^{s/a} = O(|\alpha|^{-1})$ and $F_+^{s/a} = O(|\alpha|^{-1/2})$ as $|\alpha| \to \infty$. Hence, all terms in Equation (4.3) tend to zero as $|\alpha| \to \infty$. On applying Liouville's theorem, $J^{s/a}(\alpha)$ must therefore be identically zero, and so

$$F_-^{s/a}(\alpha) = H_-^{s/a}(\alpha) K_-^{s/a}(\alpha) = \frac{-k}{\varepsilon - \mathrm{i}\alpha} + \frac{k\, K_-^{s/a}(\alpha)}{K_-^{s/a}(-\mathrm{i}\varepsilon)(\varepsilon - \mathrm{i}\alpha)}\,, \tag{4.25}$$

$$F_+^{s/a}(\alpha) = \frac{H_+^{s/a}(\alpha)}{K_+^{s/a}(\alpha)} = -\frac{k}{(\varepsilon - \mathrm{i}\alpha)\, K_+^{s/a}(\alpha)\, K_-^{s/a}(-\mathrm{i}\varepsilon)}\,. \tag{4.26}$$

Following the steps described at the end of Section 3, we obtain, after straightforward calculations,

$$2A_1^s = \frac{k\, K_-^s(\alpha)}{K_-^s(-\mathrm{i}\varepsilon)\,(\varepsilon - \mathrm{i}\alpha)\cosh(\alpha h)}\,, \tag{4.27}$$

$$-2A_1^a = \frac{k\, K_-^a(\alpha)}{K_-^a(-\mathrm{i}\varepsilon)\,(\varepsilon - \mathrm{i}\alpha)\sinh(\alpha h)}\,, \tag{4.28}$$

$$\varphi^{s/a}(x, y) = \begin{cases} \dfrac{k\, c_2^{s/a}}{\sqrt{2\pi}\, c_5^{s/a}} \displaystyle\int_{-\infty}^{\infty} \Phi_i^{s/a}(\alpha, x, y)\, \mathrm{d}\alpha\,, & 0 \le y < h\,, \\[2ex] \dfrac{k\, c_2^{s/a}}{\sqrt{2\pi}\, c_5^{s/a}} \displaystyle\int_{-\infty}^{\infty} \Phi_o^{s/a}(\alpha, x, y)\, \mathrm{d}\alpha\,, & h \le y \le L\,, \end{cases} \tag{4.29}$$

where

$$\Phi_i^s(\alpha, x, y) = \frac{\Gamma(1/2)}{(\varepsilon - \mathrm{i}\alpha)\, K_-^s(-\mathrm{i}\varepsilon)} \frac{\displaystyle\prod_{l=1}^{2p} \Gamma\left[\frac{\mathrm{i}\alpha L}{2p\pi} + \frac{\delta_l L}{2p\pi}\right]}{\Gamma\left[1 + \mathrm{i}\dfrac{\alpha(L-h)}{\pi}\right]} \times \frac{\Gamma\left(\frac{1}{2} - \mathrm{i}\dfrac{\alpha h}{\pi}\right)}{\pi} \cosh(\alpha y)\, \mathrm{e}^{-\mathrm{i}\alpha(x + \log\, c_4)}\,, \tag{4.30}$$

$$\Phi_o^s(\alpha,x,y) = \frac{\Gamma(1/2)}{(\varepsilon-\mathrm{i}\alpha)\,K_-^s(-\mathrm{i}\varepsilon)}\,\frac{\prod_{l=1}^{2p}\Gamma\left[\frac{\mathrm{i}\alpha L}{2p\pi}+\frac{\delta_l L}{2p\pi}\right]}{\Gamma\left[\frac{1}{2}+\mathrm{i}\frac{\alpha h}{\pi}\right]}\times$$

$$\frac{\Gamma\left(1-\mathrm{i}\frac{\alpha(L-h)}{\pi}\right)}{\alpha(L-h)}\sinh\left(\alpha(L-y)\right)\,\mathrm{e}^{-\mathrm{i}\alpha(x+\log c_4)}\,, \tag{4.31}$$

$$\Phi_i^a(\alpha,x,y) = \frac{1}{(\varepsilon-\mathrm{i}\alpha)\,K_-^a(-\mathrm{i}\varepsilon)}\,\frac{\prod_{l=1}^{2p}\Gamma\left[\frac{\mathrm{i}\alpha L}{2p\pi}+\frac{\gamma_l L}{2p\pi}\right]}{\Gamma\left[1+\mathrm{i}\frac{\alpha(L-h)}{\pi}\right]}\times$$

$$\frac{\Gamma\left(1-\mathrm{i}\frac{\alpha h}{\pi}\right)}{\alpha h}\sinh\left(\alpha y\right)\,\mathrm{e}^{-\mathrm{i}\alpha(x+\log c_4)}\,, \tag{4.32}$$

$$\Phi_o^a(\alpha,x,y) = \frac{1}{(\varepsilon-\mathrm{i}\alpha)\,K_-^a(-\mathrm{i}\varepsilon)}\,\frac{\prod_{l=1}^{2p}\Gamma\left[\frac{\mathrm{i}\alpha L}{2p\pi}+\frac{\gamma_l L}{2p\pi}\right]}{\Gamma\left[1+\mathrm{i}\frac{\alpha h}{\pi}\right]}\times$$

$$\frac{\Gamma\left(1-\mathrm{i}\frac{\alpha(L-h)}{\pi}\right)}{\alpha(L-h)}\sinh\left(\alpha(L-y)\right)\,\mathrm{e}^{-\mathrm{i}\alpha(x+\log c_4)}\,. \tag{4.33}$$

The indices 'i' and 'o' are labels for 'inside' and 'outside' the channel. The integrals can be evaluated using the residue theorem (see [10]). We close the contour by a semicircle CR of radius R and center at $\alpha = 0$. The contour is closed in the lower half-plane when $x > 0$ and in the upper half-plane when $x < 0$. The contribution of the semicircle to the integrals vanishes when R tends to infinity since K_- is asymptotic to $|\alpha|^{-1/2}$ as $|\alpha| \to \infty$. We employ the following:

$$\varphi^{s/a}(x,y) = \begin{cases} \varphi_1^{s/a}(x,y)\,, & 0 \le y \le h\,,\ 0 \le x\,,\\ \varphi_2^{s/a}(x,y)\,, & 0 \le y \le h\,,\ x \le 0\,,\\ \varphi_3^{s/a}(x,y)\,, & h < y \le L\,,\ x \le 0\,,\\ \varphi_4^{s/a}(x,y)\,, & h < y \le L\,,\ 0 \le x\,. \end{cases} \tag{4.34}$$

Straightforward calculations and invoking the limit $\varepsilon \to 0$ yield

$$\varphi_1^s(x, y) = V \sum_{n=1}^{\infty} \mathbf{a^s_{1n}} \cos(\lambda_n y) \mathrm{e}^{-\lambda_n x} + V , \tag{4.35}$$

$$\varphi_2^s(x, y) = V \sum_{n=1}^{\infty} \sum_{l=1}^{2p} \mathbf{a^s_{2\,nl}} \cos(\delta_{nl}\, y)\, \mathrm{e}^{\delta_{nl}\, x} , \tag{4.36}$$

$$\varphi_3^s(x, y) = V \sum_{n=1}^{\infty} \sum_{l=1}^{2p} \mathbf{a^s_{3\,nl}} \sin(\delta_{nl}\,(L-y))\, \mathrm{e}^{\delta_{nl}\, x} , \tag{4.37}$$

$$\varphi_4^s(x, y) = V \sum_{n=1}^{\infty} \mathbf{a^s_{4n}} \sin(\kappa_n (L-y)) \mathrm{e}^{-\kappa_n x} + \frac{V(L-y)}{L-h} , \tag{4.38}$$

where

$$\mathbf{a^s_{1n}} = \frac{1}{c_5^s} \frac{\prod_{j=1}^{2p} G_j^s(\lambda_n)}{\Gamma(1+\lambda_n \frac{L-h}{\pi})} \frac{(-1)^n \sqrt{\pi}}{\lambda_n\,(n-1)!\,h}\, \mathrm{e}^{-\lambda_n \log c_4} , \tag{4.39}$$

$$\mathbf{a^s_{2nl}} = \frac{1}{c_5^s} \frac{\prod_{j=1}^{l-1} G_j^s(-\delta_{nl}) \prod_{j=l+1}^{2p} G^s(-\delta_{nl})\, \Gamma\left[1/2+\delta_{nl}\frac{h}{\pi}\right]}{\Gamma\left(1-\delta_{nl}\frac{L-h}{\pi}\right)} \times \frac{(-1)^{n-1}\sqrt{\pi}\,2p}{\delta_{nl}\,(n-1)!\,L}\, \mathrm{e}^{\delta_{nl} \log c_4} , \tag{4.40}$$

$$\mathbf{a^s_{3nl}} = \frac{1}{c_5^s} \frac{\prod_{j=1}^{l-1} G_j^s(-\delta_{nl}) \prod_{j=l+1}^{2p} G^s(-\delta_{nl})\, \Gamma\left[1+\delta_{nl}\frac{L-h}{\pi}\right]}{\Gamma\left(1/2-\delta_{nl}\frac{h}{\pi}\right)} \times$$

$$frac(-1)^{n-1}\sqrt{\pi}\,\pi\,2p\delta_{nl}^2\,(n-1)!\,L\,(L-h)\,\mathrm{e}^{\delta_{nl}\log c_4} , \tag{4.41}$$

$$\mathbf{a^s_{4n}} = \frac{1}{c_5^s} \frac{\prod_{j=1}^{2p} G_j^s(\kappa_n)}{\Gamma(1/2+\kappa_n \frac{h}{\pi})} \frac{(-1)^n \sqrt{\pi}}{\kappa_n\, n!\,(L-h)}\, \mathrm{e}^{-\kappa_n \log c_4} \tag{4.42}$$

and where $G_j^s(x) = \Gamma\left[(x+\delta_j)\,\frac{L}{2p\pi}\right]$. The solution for the anti-symmetric problem is

$$\varphi_1^a(x, y) = V \sum_{n=1}^{\infty} \mathbf{a^a_{1n}} \sin(\mu_n y) \mathrm{e}^{-\mu_n x} + \frac{Vy}{h} , \tag{4.43}$$

$$\varphi_2^a(x,y) = V \sum_{n=1}^{\infty} \sum_{l=1}^{2p} \mathbf{a^a_{2\,nl}} \sin(\gamma_{nl}\, y)\, e^{\gamma_{nl}\, x} , \tag{4.44}$$

$$\varphi_3^a(x,y) = V \sum_{n=1}^{\infty} \sum_{l=1}^{2p} \mathbf{a^a_{3\,nl}} \sin(\gamma_{nl}\,(L-y))\, e^{\gamma_{nl}\, x} , \tag{4.45}$$

$$\varphi_4^a(x,y) = V \sum_{n=1}^{\infty} \mathbf{a^a_{4n}} \sin(\kappa_n (L-y)) e^{-\kappa_n\, x} + \frac{V(L-y)}{L-h} , \tag{4.46}$$

where

$$\mathbf{a^a_{1n}} = \frac{1}{c_5^a} \frac{\prod_{j=1}^{2p} G_j^a(\mu_n)}{\Gamma(1+\mu_n \frac{L-h}{\pi})} \frac{(-1)^n}{\mu_n\, n!\, h} e^{-\mu_n \log c_4} , \tag{4.47}$$

$$\mathbf{a^a_{2nl}} = \frac{1}{c_5^a} \frac{\prod_{j=1}^{l-1} G_j^a(-\gamma_{nl}) \prod_{j=l+1}^{2p} G^a(-\gamma_{nl})\, \Gamma\left[1+\gamma_{nl}\frac{h}{\pi}\right]}{\Gamma\left(1-\gamma_{nl}\frac{L-h}{\pi}\right)} \times \frac{(-1)^{n-1}\,\pi\, 2p}{\gamma_{nl}^2\,(n-1)!\, L\, h} e^{\gamma_{nl} \log c_4} , \tag{4.48}$$

$$\mathbf{a^a_{3nl}} = \frac{1}{c_5^a} \frac{\prod_{j=1}^{l-1} G_j^a(-\gamma_{nl}) \prod_{j=l+1}^{2p} G^a(-\gamma_{nl})\, \Gamma\left[1+\gamma_{nl}\frac{L-h}{\pi}\right]}{\Gamma\left(1-\gamma_{nl}\frac{h}{\pi}\right)} \times$$

$$\frac{(-1)^{n-1}\,\pi\, 2p}{\gamma_{nl}^2\,(n-1)!\, L\,(L-h)} e^{\gamma_{nl} \log c_4} , \tag{4.49}$$

$$\mathbf{a^a_{4n}} = \frac{1}{c_5^a} \frac{\prod_{j=1}^{2p} G_j^a(\kappa_n)}{\Gamma(1+\kappa_n \frac{h}{\pi})} \frac{(-1)^n}{\kappa_n\, n!\,(L-h)} e^{-\kappa_n \log c_4} , \tag{4.50}$$

and where $G_j^a(x) = \Gamma\left[(x+\gamma_j)\frac{L}{2p\pi}\right]$.

The solution of (2.1–2.8) can be written as

$$\varphi(x,y) = \begin{cases} \varphi_1(x,y) , & |\,y\,| \le h\,,\ 0 \le x\,, \\ \varphi_2(x,y) , & |\,y\,| \le h\,,\ x \le 0\,, \\ \varphi_3(x,y) , & h < |\,y\,| \le L\,,\ x \le 0\,, \\ \varphi_4(x,y) , & h < |\,y\,| \le L\,,\ 0 \le x\,, \end{cases} \tag{4.51}$$

where

$$\varphi_i(x,y) = \varphi_i^s(x,|\,y\,|) + \mathrm{sign}(y)\, \varphi_i^a(x,|\,y\,|) \tag{4.52}$$

for $i = 1, 2, 3, 4$.

In the numerical representation of the solution the infinite sums from (4.35–4.38) and (4.43–4.46) will be truncated after the Nth term. If the truncation number for $\varphi_2^{s/a}$ and $\varphi_3^{s/a}$ is N_1, it is better to choose $N_2 = 2p\,N_1$ as the value for the truncation number for $\varphi_1^{s/a}$ and $\varphi_4^{s/a}$. This choice is motivated by the necessity of having the same precision of the solution (which is given by the number of terms in the sums) for $x < 0$ and $x > 0$.

5. Electric field near the electrode edges

In order to determine the behaviour of the electric field at the ends of the electrodes, we need to evaluate the quantities $\varphi_{,x}^{s/a}$ and $\varphi_{,y}^{s/a}$ when $(x, y) \to (0, h^{\pm})$. Differentiating and using the properties of the Fourier transform, we have

$$\varphi_{,x}^{s/a}(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-\mathrm{i})\alpha\,\Phi^{s/a}(\alpha, y)\,\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}\alpha\ , \tag{5.1}$$

$$\varphi_{,y}^{s/a}(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_{,y}^{s/a}(\alpha, y)\,\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}\alpha\ , \tag{5.2}$$

where $\Phi^{s/a}(\alpha, y)$ are found by inserting (4.27) and (4.28) in (3.14) and (3.18), respectively. Let us first consider the asymptotic evaluation of $\varphi_{,x}^{s/a}$ when $(x, y) \to (0, h^-)$. The integral can be written as the sum

$$\varphi_{,x}^{s/a}(x, y) = \int_{-\infty}^{-M} E(\alpha, x, y)\,\mathrm{d}\alpha + \int_{-M}^{M} E(\alpha, x, y)\,\mathrm{d}\alpha + \int_{M}^{\infty} E(\alpha, x, y)\,\mathrm{d}\alpha\ , \tag{5.3}$$

where $M > 0$ and $E(\alpha, x, y) = \dfrac{k\,K_-^s(\alpha)\cosh(\alpha y)(-\mathrm{i})\alpha}{\sqrt{2\pi}K_-^s(-\mathrm{i}\varepsilon)\,(\varepsilon - \mathrm{i}\alpha)\cosh(\alpha h)(\varepsilon - \mathrm{i}\alpha)}\,\mathrm{e}^{-\mathrm{i}\alpha x}$. Assuming that M is sufficiently large, we can replace E in the first and third integrals by its asymptotic expression valid for large arguments. The second term can be neglected because E is a bounded function of α on the interval $[-M, M]$ and, consequently, $\int_{-M}^{M} E\,\mathrm{d}\alpha$ will be a continuous function of x. By using (4.18) and $\cosh(\alpha y)/\cosh(\alpha h) \sim \mathrm{e}^{|\alpha|(y-h)}$ when $\alpha \to \pm\infty$, we obtain

$$\varphi_{,x}^{s} \sim \frac{k\,B^s}{\sqrt{2\pi}\,K_-^s(-\mathrm{i}\varepsilon)} \left[\int_{-\infty}^{-M} \frac{\mathrm{e}^{\mathrm{i}\alpha x + \alpha(y-h)}}{\sqrt{-\mathrm{i}\alpha}}\,\mathrm{d}\alpha + \int_{M}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\alpha x + \alpha(y-h)}}{\sqrt{\mathrm{i}\alpha}}\,\mathrm{d}\alpha \right]\ , \tag{5.4}$$

as $(x, y) \to (0, h^-)$. If we take the limit $M \to 0$, we may ignore the resulting finite contributions because they do not alter the singular behaviour. Using the formula [9, pp. 255]

$$\int_0^{\infty} \mathrm{e}^{-t\alpha}\,\alpha^{-1/2}\mathrm{d}\alpha = \sqrt{\frac{\pi}{t}}\ , \quad \mathfrak{Re}\,t > 0\ , \tag{5.5}$$

in (5.4), yields

$$\varphi^{s}_{,x} \sim \frac{\sqrt{2}\,k\,B^{s}}{K^{s}_{-}(-\mathrm{i}\varepsilon)}\,\frac{\sin(\theta/2)}{\sqrt{r}}\,, \quad \text{where} \quad r\mathrm{e}^{\mathrm{i}\theta} = x + \mathrm{i}(y-h)\,. \tag{5.6}$$

The other cases are treated analogously and similar expressions are derived. After applying the limit $\varepsilon \to 0$, we readily find

$$\varphi^{s/a}_{,x} \sim \frac{\sqrt{2}\,k\,B^{s/a}}{c_2^{s/a}}\,\frac{\sin(\theta/2)}{\sqrt{r}}\,, \quad \varphi^{s/a}_{,y} \sim \frac{\sqrt{2}\,k\,B^{s/a}}{c_2^{s/a}}\,\frac{\cos(\theta/2)}{\sqrt{r}}\,, \tag{5.7}$$

when $(x, y) \to (0, h^{\pm})$ and where $x \pm \mathrm{i}(y-h) = r\mathrm{e}^{\mathrm{i}\theta}$. The singular behaviour of the electric field

$$E^{s/a}(x, y) = \sqrt{(\varphi^{s/a}_{,x})^2 + (\varphi^{s/a}_{,y})^2} \tag{5.8}$$

is then given by

$$E^{s/a}(x, y) \sim \frac{\sqrt{2}\,k\,B^{s/a}}{c_2^{s/a}}\,\frac{1}{\sqrt{r}} \quad , r = \sqrt{(x^2+y^2)}\,, \tag{5.9}$$

when $(x, y) \to (0, h^{\pm})$. Thus, we have established square-root singularities of the electric field at the electrode tips.

6. Results and discussion

The solution given at the end of Section 4 has a high degree of generality. First, it can be applied to different configurations of the electrodes (see Table 1). Each of these configurations can be extended to the case of finite electrodes in which the far edges of the electrodes do not interact. To determine the appropriate length of the electrodes one can use the following criterion

$$|E_x(x, y)| < 0{\cdot}05\, E_\infty(y) \quad \text{for} \quad x > a\,, \ \ |y| \le L \tag{6.1}$$

where a is the half length of the electrodes, $E_x(x, y) = \varphi_{,x}(x, y)$ and $E_\infty(y) = \lim_{x\to\infty} \sqrt{(\varphi_{,x}(x, y))^2 + (\varphi_{,y}(x, y))^2}$. This means that the electric field becomes approximately uniform for $x > a$. For instance, a parametric study done for an anti-symmetric configuration, for values of L, h and c so that $2 \le L/h \le 100$ and $0{\cdot}01 < c < 50$, shows that (6.1) holds for $x \ge L - h$ outside the channel and for $x \ge 0{\cdot}8\,h$ inside the channel. Consequently, one can build the solution for the case of two finite electrodes of length $2\,a$, with $a > L - h$, charged in an anti-symmetric way, by taking the solution (4.43–4.46) for $x < a$ and extending it by symmetry with respect to the line $x = a$. Similar studies can be done for each case, and they are very useful for the numerical modeling.

Second, the solution depends on parameters like L/h and c, each of these having a certain influence on the profile of the electric potential. However, this section will not contain a detailed discussion of all these cases. We consider it more reasonable to do this in connection with the investigations of the flow in order to determine the optimal parameters for the desired effects on the fluid flow. We limit ourselves only to stating a number of configurations in

Table 1. Schematic diagrams of the configurations for which the solution is applicable

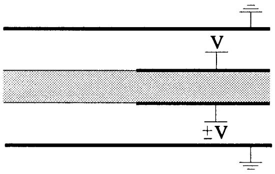
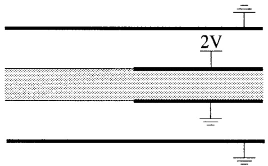
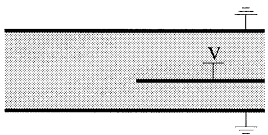
	Two semi-infinite electrodes symmetrically or anti-symmetrically charged along the channel walls and two infinite grounded electrodes
	Two semi-infinite electrodes non-symmetrically charged along the channel walls and two infinite grounded electrodes
	One semi-infinite electrode placed inside the fluid flowing through a grounded channel

Table 1, offering a few comments and postponing further studies of the solution to another paper (in preparation) in which we will examine the influence of the parameters on the channel flow of an electrorheological fluid.

In all the graphical representations that follow we use the dimensionless quantities

$$\tilde{x} = \frac{x}{h}\,, \quad \tilde{y} = \frac{y}{h}\,, \quad \tilde{L} = \frac{L}{h}\,, \quad \tilde{h} = 1\,, \quad \tilde{\varphi}(\tilde{x}, \tilde{y}) = \frac{\varphi(x, y)}{V}\,, \quad \tilde{E}(\tilde{x}, \tilde{y}) = E(x, y)\frac{h}{V}\,. \quad (6.2)$$

All plots are done for the case of semi-infinite electrodes and we use the truncation numbers $N_1 = 100$ and $N_2 = 2p\,N_1$.

We note that the electric permittivities ε_1 and ε_2 do not appear explicitly in the solution. Only their ratio c influences the results through γ_{nl} and δ_{nl}, which are computed numerically from Equations (4.6) and (4.7). Since the ERF (Rheobay for example) can exhibit values of the electric permittivity around 10^{-9} As/Vm (see [11, p. 53]) which is quite large, it is reasonable to consider small values of the ratio c and we take $c = 0{\cdot}02$ as a usual technical value. Nevertheless, we show in Figure 3 what effects are produced by different values of c on the electric potential and, consequently, on the electric field. The values of the potential are equidistant, every second contour being marked. The contours are shifted upwards and the non-symmetry becomes more pronounced as c is larger. If $c = 0{\cdot}02$, the electric field in the channel is decreasing faster when x tends to $-\infty$ than if $c = 50$. As expected, for $c = 0{\cdot}02$, the electric field outside the channel is larger than inside and this is reversed for $c = 50$. In particular, this means that the insulator surrounding the channel has a significant influence on the flow behaviour of the ERF inside the channel.

Finally, besides the non-symmetric case we want to refer to the other cases to which our results can be applied (see Table 1). One can study the electric field also in the anti-symmetric case. Here the potential in the middle of the channel vanishes. Therefore, one

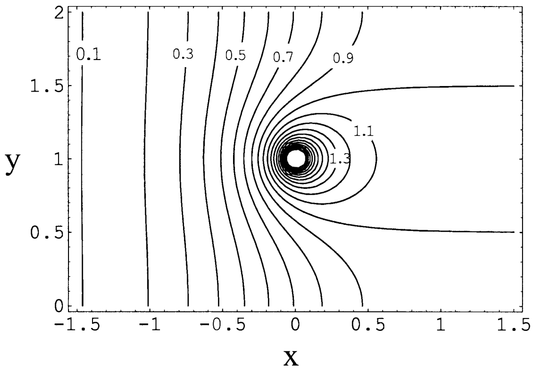


Figure 4. Contour lines of the electric field modulus, $\tilde{E}(\tilde{x}, \tilde{y})$, around the edge of a single electrode placed inside the fluid; the channel walls are placed at $\tilde{y} = 0$ and $\tilde{y} = \tilde{L}$ and they are grounded ($c = 1$, $\tilde{L} = 2\tilde{h}$).

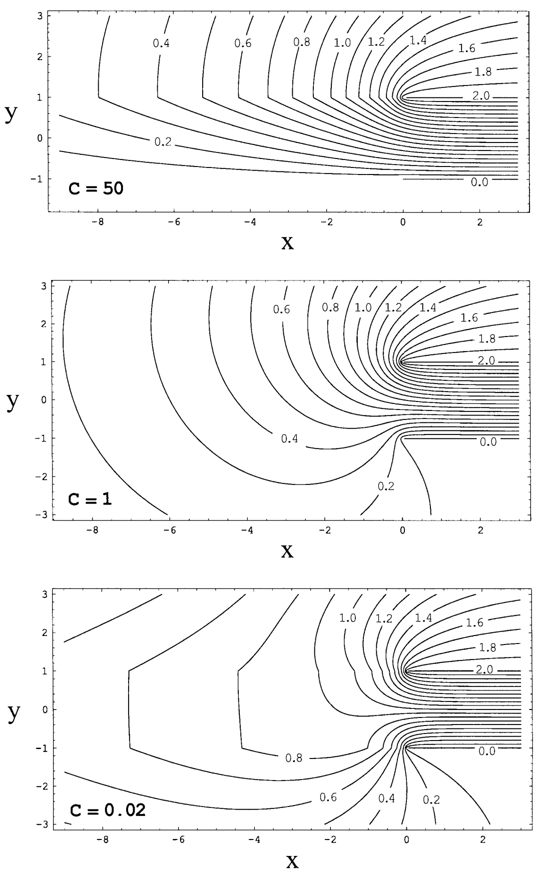


Figure 3. Equipotential lines $\tilde{\varphi}(\tilde{x}, \tilde{y}) = \text{constant}$ for different values of $c = \varepsilon_1/\varepsilon_2$ ($\tilde{L} = 10$).

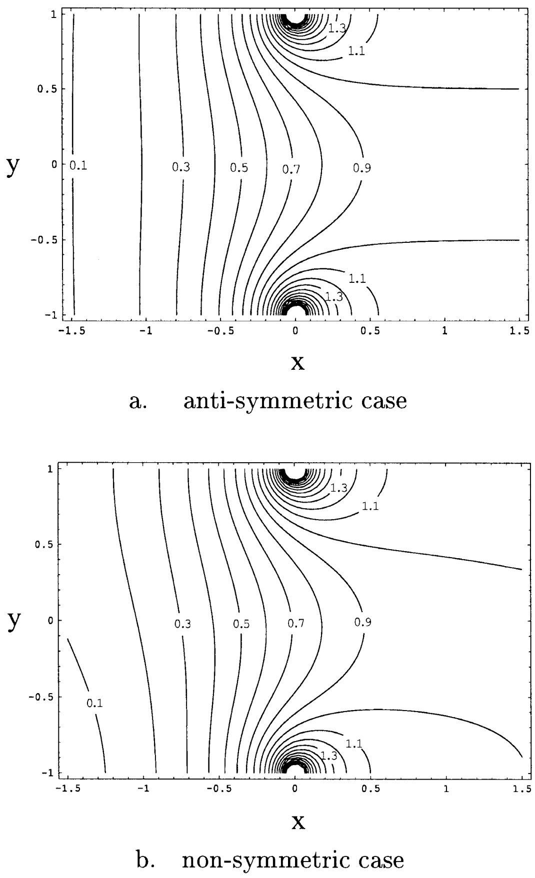


Figure 5. Contour lines of $\tilde{E}(\tilde{x}, \tilde{y})$ inside the channel, in the anti-symmetric and non-symmetric configurations ($c = 0{\cdot}02$, $\tilde{L} = 10$).

can use the solution (4.43–4.50) with $c = 1$ to characterize the case of a single electrode, placed inside the fluid, parallel with the channel walls which are considered to be grounded. To illustrate this case we have plotted in Figure 4 the modulus of the electric field, $\tilde{E}(\tilde{x}, \tilde{y}) = \sqrt{(\frac{\partial\tilde{\varphi}}{\partial\tilde{x}}(\tilde{x}, \tilde{y}))^2 + (\frac{\partial\tilde{\varphi}}{\partial\tilde{y}}(\tilde{x}, \tilde{y}))^2}$ produced by an electrode placed at $\tilde{y} = \tilde{L}/2$. In Figure 5 the electric-field moduli, $\tilde{E}(\tilde{x}, \tilde{y})$ in the anti-symmetric case, is compared with that in the non-symmetric case for $c = 0{\cdot}02$. It can be seen that the profiles are quite similar near the electrode ends but become very different for $\tilde{x} < -0{\cdot}5$. In Figures 4 and 5 every second contour is marked, except when the lines are too close to one another. In the white regions around the points where the electrode edges are situated, the electric field is greater than 3.

7. Conclusions

In this work we have found analytically and computed numerically the electric-potential distribution in a complex configuration of electrodes. The problem arises from the study of electrorheological fluids in channel flow. Because the dielectric permittivity of the ERF is very high, a considerable jump in the y-derivatives has to be taken into account. The mixed-boundary-value problem is split up in two problems which can be solved through application of the Wiener-Hopf technique. The results can be used to describe the electric field generated between two infinite grounded electrodes by either one long electrode or two long electrodes charged in an anti-symmetric or a non-symmetric way. The solution we give here allows for the first time to model realistically an electrorheological fluid in two-dimensional channel flow. For a complete treatment of the ERF one should find how to control the effects on the fluid through the problem parameters.

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